Coupled Painlevé VI system with $E^{(1)}_6$-symmetry

2009 J. Phys. A: Math. Theor. 42 145205

(http://iopscience.iop.org/1751-8121/42/14/145205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 145205 (11pp) doi:10.1088/1751-8113/42/14/145205

Coupled Painlevé VI system with $E_6^{(1)}$-symmetry

Kenta Fuji and Takao Suzuki

Department of Mathematics, Kobe University Rokko, Kobe 657-8501, Japan

E-mail: suzukit@math.kobe-u.ac.jp

Received 9 November 2008, in final form 24 February 2009
Published 16 March 2009
Online at stacks.iop.org/JPhysA/42/145205

Abstract
We present a new system of ordinary differential equations with affine Weyl group symmetry of type $E_6^{(1)}$. This system is expressed as a Hamiltonian system of sixth order with a coupled Painlevé VI Hamiltonian.

PACS number: 02.30.Ik
Mathematics Subject Classification: 34M55, 17B80, 37K10

Introduction

The Painlevé equations $P_J(J = \mathrm{I}, \ldots, \mathrm{VI})$ are ordinary differential equations of second order. It is known that these P_J admit the following affine Weyl group symmetries [O1]:

P_{I}	P_{II}	P_{III}	P_{IV}	P_{V}	P_{VI}
–	$A_1^{(1)}$	$A_1^{(1)} \oplus A_1^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$D_4^{(1)}$

Several extensions of the Painlevé equations have been studied from the viewpoint of affine Weyl group symmetry. The Noumi–Yamada system is a generalization of P_{II}, P_{IV} and P_{V} for $A_n^{(1)}$-symmetry [NY1]. The coupled Painlevé VI system with $D_{2n+2}^{(1)}$-symmetry is also studied [S]. In this paper, we present a new system of ordinary differential equations with $E_6^{(1)}$-symmetry. Our system can be expressed as a Hamiltonian system of sixth order with a coupled Painlevé VI Hamiltonian.

In order to obtain this system, we consider a similarity reduction of a Drinfeld–Sokolov hierarchy of type $E_6^{(1)}$. The Drinfeld–Sokolov hierarchies are extensions of the KdV (or mKdV) hierarchy [DS]. They are characterized by graded Heisenberg subalgebras of affine Lie algebras. They also imply several Painlevé systems by similarity reduction as follows [AS, FS1, FS2, KIK, KK1, KK2]:

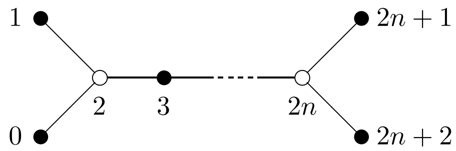

Figure 1. Gradation of $\mathfrak{g}(D^{(1)}_{2n+2})$ of type $(1, 1, 0, 1, 0, \ldots, 1, 0, 1, 1)$.

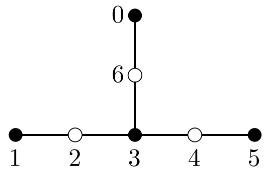

Figure 2. Gradation of $\mathfrak{g}(E^{(1)}_6)$ of type $(1, 1, 0, 1, 0, 1, 0)$.

Lie algebra	Gradation	Painlevé system
$A^{(1)}_1$	$(1, 1)$ $(1, 0)$	$P_{\rm II}$ $P_{\rm IV}$
$A^{(1)}_2$	$(1, 1, 1)$ $(2, 1, 1)$ $(1, 0, 0)$	$P_{\rm IV}$ $P_{\rm V}$ $P_{\rm VI}$
$A^{(1)}_3$	$(1, 1, 1, 1)$	$P_{\rm V}$
$A^{(1)}_n (n \geqslant 4)$	$(1, \ldots, 1)$	Noumi–Yamada system
$D^{(1)}_4$	$(1, 1, 0, 1, 1)$	$P_{\rm VI}$
$D^{(1)}_{2n+2} (n \geqslant 2)$	$(1, 1, 0, 1, 0, \ldots, 1, 0, 1, 1)$	Coupled $P_{\rm VI}$

.

As is seen above, the coupled Painlevé VI system is derived from the $D^{(1)}_{2n+2}$-hierarchy associated with the graded Heisenberg subalgebra of type $(1, 1, 0, 1, 0, \ldots, 1, 0, 1, 1)$. We apply a similar method to the case of $E^{(1)}_6$ by choosing the graded Heisenberg subalgebra of type $(1, 1, 0, 1, 0, 1, 0)$; see figures 1 and 2. The hierarchy defined thus implies our new system by similarity reduction.

This paper is organized as follows. In section 1, we present an explicit formula of a coupled Painlevé VI system with $E^{(1)}_6$-symmetry. In section 2, we recall the affine Lie algebra $\mathfrak{g}(E^{(1)}_6)$ and its graded Heisenberg subalgebra of type $(1, 1, 0, 1, 0, 1, 0)$. In section 3, we formulate a similarity reduction of a Drinfeld–Sokolov hierarchy of type $E^{(1)}_6$. In section 4, we derive the coupled Painlevé VI system from the similarity reduction.

1. Main result

The Painlevé equation $P_{\rm VI}$ can be expressed as the following Hamiltonian system [IKSY, O2]:

$$s(s-1)\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\partial H_{\rm VI}}{\partial p}, \qquad s(s-1)\frac{\mathrm{d}p}{\mathrm{d}s} = -\frac{\partial H_{\rm VI}}{\partial q},$$

with the Hamiltonian $H_{\mathrm{VI}} = H_{\mathrm{VI}}(p, q, s; \beta_0, \beta_1, \beta_3, \beta_4)$ defined by

$$H_{\mathrm{VI}} = q(q-1)(q-s)p^2 - \{(\beta_1 - 1)q(q-1) + \beta_3 q(q-s) + \beta_4(q-1)(q-s)\}p + \beta_2(\beta_0 + \beta_2)q,$$

where $\beta_i (i = 0, \dots, 4)$ are complex parameters satisfying

$$\beta_0 + \beta_1 + 2\beta_2 + \beta_3 + \beta_4 = 1.$$

We define a *coupled* Hamiltonian H by

$$\begin{aligned} H = {} & H_{\mathrm{VI}}(p_1, q_1, s; \alpha_3, 1 - \alpha_1 - 2\alpha_2 - 2\alpha_3, \alpha_1, \alpha_3) \\ & + H_{\mathrm{VI}}(p_2, q_2, s; \alpha_3, 1 - 2\alpha_3 - 2\alpha_4 - \alpha_5, \alpha_5, \alpha_3) \\ & + H_{\mathrm{VI}}(p_3, q_3, s; \alpha_3, 1 - \alpha_0 - 2\alpha_3 - 2\alpha_6, \alpha_0, \alpha_3) \\ & + \sum_{1 \leqslant i < j \leqslant 3} \{(q_i - 1)p_i + \alpha_{2i}\}\{(q_j - 1)p_j + \alpha_{2j}\}(q_i q_j + s), \end{aligned} \tag{1.1}$$

where $\alpha_i (i = 0, \dots, 6)$ are complex parameters satisfying

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 = 1.$$

Note that these parameters correspond to the simple roots of type $E_6^{(1)}$. We consider a Hamiltonian system with the Hamiltonian (1.1),

$$s(s-1)\frac{\mathrm{d}q_i}{\mathrm{d}s} = \{H, q_i\}, \qquad s(s-1)\frac{\mathrm{d}p_i}{\mathrm{d}s} = \{H, p_i\} \quad (i = 1, 2, 3), \tag{1.2}$$

where $\{\cdot, \cdot\}$ stands for the Poisson bracket defined by

$$\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \quad (i, j = 1, 2, 3).$$

The affine Weyl group $W(E_6^{(1)})$ is generated by the transformations $r_i (i = 0, \dots, 6)$ acting on the simple roots as

$$r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \quad (i, j = 0, \dots, 6),$$

where $A = (a_{ij})_{i,j=0}^6$ is the generalized Cartan matrix of type $E_6^{(1)}$ defined by

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}.$$

Let $\pi_i (i = 1, 2)$ be Dynkin diagram automorphisms acting on the simple roots as

$$\pi_i(\alpha_j) = \alpha_{\sigma_i(j)} \quad (i = 1, 2; j = 0, \dots, 6),$$

where $\sigma_i (i = 1, 2)$ are permutations defined by

$$\sigma_1 = (01)(26), \qquad \sigma_2 = (05)(46).$$

We consider an extension of $W(E_6^{(1)})$

$$\widetilde{W} = \langle r_0, r_1, r_2, r_3, r_4, r_5, r_6, \pi_1, \pi_2 \rangle,$$

with the fundamental relations

$$r_i^2 = 1 \qquad (i = 0, \dots, 6),$$
$$(r_i r_j)^{2-a_{ij}} = 0 \qquad (i, j = 0, \dots, 6; i \neq j),$$
$$\pi_i^2 = 1 \qquad (i = 1, 2),$$
$$(\pi_1\pi_2)^3 = 1,$$
$$\pi_i r_j = r_{\sigma_i(j)}\pi_i \qquad (i = 1, 2; j = 0, \dots, 6).$$

The action of the group $\widetilde{W}$ can be lifted to canonical transformations of the Hamiltonian system (1.2). Denoting by

$$\varphi_0 = q_3 - 1, \quad \varphi_1 = q_1 - 1, \quad \varphi_2 = p_1, \quad \varphi_3 = q_1q_2q_3 - s,$$
$$\varphi_4 = p_2, \qquad \varphi_5 = q_2 - 1, \quad \varphi_6 = p_3,$$

we obtain

Theorem 1.1. *The system (1.2) with (1.1) is invariant under the action of birational canonical transformations* $r_i (i = 0, \dots, 6)$ *and* $\pi_i (i = 1, 2)$ *defined by*

$$r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, \quad r_i(\varphi_j) = \varphi_j + \frac{\alpha_i}{\varphi_i}\{\varphi_i, \varphi_j\} \quad (i, j = 0, \dots, 6),$$

and

$$\pi_i(\alpha_j) = \alpha_{\sigma_i(j)}, \quad \pi_i(\varphi_j) = \varphi_{\sigma_i(j)} \quad (i = 1, 2; j = 0, \dots, 6).$$

2. Affine Lie algebra

Following the notation of [Kac], we recall the affine Lie algebra $\mathfrak{g} = \mathfrak{g}\big(E_6^{(1)}\big)$ and its graded Heisenberg subalgebra of type (1, 1, 0, 1, 0, 1, 0).

The affine Lie algebra $\mathfrak{g}$ is generated by the Chevalley generators $e_i, f_i, \alpha_i^\vee (i = 0, \dots, 6)$ and the scaling element d with the fundamental relations

$$(\mathrm{ad}e_i)^{1-a_{ij}}(e_j) = 0, \quad (\mathrm{ad}f_i)^{1-a_{ij}}(f_j) = 0 \quad (i \neq j),$$
$$\big[\alpha_i^\vee, \alpha_j^\vee\big] = 0, \quad \big[\alpha_i^\vee, e_j\big] = a_{ij}e_j, \quad \big[\alpha_i^\vee, f_j\big] = -a_{ij}f_j, \quad [e_i, f_j] = \delta_{i,j}\alpha_i^\vee,$$
$$\big[d, \alpha_i^\vee\big] = 0, \quad [d, e_i] = \delta_{i,0}e_0, \quad [d, f_i] = -\delta_{i,0}f_0,$$

for $i, j = 0, \dots, 6$. We denote the Cartan subalgebra of $\mathfrak{g}$ by

$$\mathfrak{h} = \bigoplus_{j=0}^{6}\mathbb{C}\alpha_j^\vee \oplus \mathbb{C}d.$$

The canonical central element of $\mathfrak{g}$ is given by

$$K = \alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee + \alpha_5^\vee + 2\alpha_6^\vee.$$

The normalized invariant form $(|) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is determined by the conditions

$$\big(\alpha_i^\vee\big|\alpha_j^\vee\big) = a_{ij}, \quad (e_i|f_j) = \delta_{i,j}, \quad \big(\alpha_i^\vee\big|e_j\big) = \big(\alpha_i^\vee\big|f_j\big) = 0,$$
$$(d|d) = 0, \qquad \big(d\big|\alpha_j^\vee\big) = \delta_{0,j}, \quad (d|e_j) = (d|f_j) = 0,$$

for $i, j = 0, \dots, 6$.

Consider the gradation $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}}\mathfrak{g}_k$ of type (1, 1, 0, 1, 0, 1, 0) by setting

$$\deg \mathfrak{h} = \deg e_i = \deg f_i = 0 \quad (i = 2, 4, 6),$$
$$\deg e_i = 1, \quad \deg f_i = -1 \qquad (i = 0, 1, 3, 5).$$

With an element $\vartheta \in \mathfrak{h}$ such that

$$\left(\vartheta \middle| \alpha_i^\vee\right) = 0 \quad (i = 2, 4, 6),$$
$$\left(\vartheta \middle| \alpha_i^\vee\right) = 1 \quad (i = 0, 1, 3, 5),$$

this gradation is defined by

$$\mathfrak{g}_k = \{x \in \mathfrak{g} \,|\, [\vartheta, x] = kx\} \quad (k \in \mathbb{Z}).$$

Note that ϑ is given explicitly by

$$\vartheta = 6d + 4\alpha_1^\vee + 7\alpha_2^\vee + 10\alpha_3^\vee + 7\alpha_4^\vee + 4\alpha_5^\vee + 5\alpha_6^\vee.$$

We denote by

$$\mathfrak{g}_{<0} = \bigoplus_{k<0} \mathfrak{g}_k, \qquad \mathfrak{g}_{\geqslant 0} = \bigoplus_{k\geqslant 0} \mathfrak{g}_k.$$

Such gradation implies the Heisenberg subalgebra of $\mathfrak{g}$

$$\mathfrak{s} = \{x \in \mathfrak{g} \,|\, [x, \Lambda_1] = \mathbb{C}K\},$$

with an element of $\mathfrak{g}_1$

$$\Lambda_1 = e_1 + 2e_3 + e_5 + e_{21} + e_{60} + e_{23} + e_{43} + e_{63} + e_{234} + e_{236} + e_{436} + 2e_{6234},$$

where

$$e_{i_1 i_2,\ldots,i_n j} = \mathrm{ad}e_{i_1}\mathrm{ad}e_{i_2}, \ldots, \mathrm{ad}e_{i_n}(e_j).$$

Note that $\mathfrak{s}$ admits the gradation of type $(1, 1, 0, 1, 0, 1, 0)$, namely

$$\mathfrak{s} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{s}_k, \quad \mathfrak{s}_k \subset \mathfrak{g}_k.$$

We also remark that the positive part of $\mathfrak{s}$ has a graded base $\{\Lambda_k\}_{k=1}^{\infty}$ satisfying

$$[\Lambda_k, \Lambda_l] = 0, \quad [\vartheta, \Lambda_k] = n_k \Lambda_k \quad (k, l = 1, 2, \ldots),$$

where n_k stands for the degree of element Λ_k defined by

$$n_{6l+1} = 6l + 1, \quad n_{6l+2} = 6l + 1, \quad n_{6l+3} = 6l + 2,$$
$$n_{6l+4} = 6l + 4, \quad n_{6l+5} = 6l + 5, \quad n_{6l+6} = 6l + 5.$$

We formulate the Drinfeld–Sokolov hierarchy of type $E_6^{(1)}$ associated with the Heisenberg subalgebra $\mathfrak{s}$ by using these Λ_k in the following section.

Remark 2.1. The isomorphism classes of the Heisenberg subalgebras are in one-to-one correspondence with the conjugacy classes of the finite Weyl group [KP]. In the notation of [C], the Heisenberg subalgebra $\mathfrak{s}$ introduced above corresponds to the regular primitive conjugacy class $E_6(a_2)$ of the Weyl group $W(E_6)$; see [DF].

3. Drinfeld–Sokolov hierarchy

In this section, we formulate a similarity reduction of a Drinfeld–Sokolov hierarchy of type $E_6^{(1)}$ associated with the Heisenberg subalgebra $\mathfrak{s}$.

In the following, we use the notation of infinite dimensional groups

$$G_{<0} = \exp(\widehat{\mathfrak{g}}_{<0}), \qquad G_{\geqslant 0} = \exp(\widehat{\mathfrak{g}}_{\geqslant 0}),$$

where $\widehat{\mathfrak{g}}_{<0}$ and $\widehat{\mathfrak{g}}_{\geqslant 0}$ are the completions of $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{\geqslant 0}$, respectively.

Let $X(0) \in G_{<0}G_{\geqslant 0}$. Introducing the time variables $t_k (k = 1, 2, \ldots)$, we consider a $G_{<0}G_{\geqslant 0}$-valued function

$$X = X(t_1, t_2, \ldots) = \exp\left(\sum_{k=1}^{\infty} t_k \Lambda_k\right) X(0).$$

Then we have a system of partial differential equations

$$X \partial_k X^{-1} = \partial_k - \Lambda_k \quad (k = 1, 2, \ldots), \tag{3.1}$$

where $\partial_k = \partial/\partial t_k$, defined through the adjoint action of $G_{<0}G_{\geqslant 0}$ on $\widehat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geqslant 0}$. Via the decomposition

$$X = W^{-1}Z, \quad W \in G_{<0}, \quad Z \in G_{\geqslant 0},$$

the system (3.1) implies a system of partial differential equations

$$\partial_k - B_k = W(\partial_k - \Lambda_k)W^{-1} \quad (k = 1, 2, \ldots), \tag{3.2}$$

where B_k stands for the $\mathfrak{g}_{\geqslant 0}$-component of $W\Lambda_k W^{-1} \in \widehat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geqslant 0}$. The Zakharov–Shabat equations,

$$[\partial_k - B_k, \partial_l - B_l] = 0 \quad (k, l = 1, 2, \ldots), \tag{3.3}$$

follow from the system (3.2).

Under the system (3.2), we consider the operator

$$\mathcal{M} = W \exp\left(\sum_{k=1}^{\infty} t_k \Lambda_k\right) \vartheta \exp\left(-\sum_{k=1}^{\infty} t_k \Lambda_k\right) W^{-1}.$$

Then the operator $\mathcal{M}$ satisfies

$$[\partial_k - B_k, \mathcal{M}] = 0 \quad (k = 1, 2, \ldots). \tag{3.4}$$

Note that

$$\mathcal{M} = W\vartheta W^{-1} - \sum_{k=1}^{\infty} n_k t_k W \Lambda_k W^{-1}.$$

Now we require that the similarity condition $\mathcal{M} \in \mathfrak{g}_{\geqslant 0}$ be satisfied. Then we have

$$\mathcal{M} = \vartheta - \sum_{k=1}^{\infty} n_k t_k B_k.$$

We also assume that $t_k = 0$ for $k \geqslant 3$. Then systems (3.3) and (3.4) are equivalent to

$$\begin{aligned} &[\partial_1 - B_1, \partial_2 - B_2] = 0, \\ &[\partial_k - B_k, \vartheta - t_1 B_1 - t_2 B_2] = 0 \quad (k = 1, 2). \end{aligned} \tag{3.5}$$

We regard the system (3.5) as a similarity reduction of the Drinfeld–Sokolov hierarchy of type $E_6^{(1)}$.

The $\mathfrak{g}_{\geqslant 0}$-valued functions $B_k (k = 1, 2)$ are expressed in the form

$$B_k = U_k + \Lambda_k, \quad U_k = \sum_{i=0}^{6} u_{k,i} \alpha_i^{\vee} + \sum_{i=2,4,6} x_{k,i} e_i + \sum_{i=2,4,6} y_{k,i} f_i.$$

In terms of the operators $U_k \in \mathfrak{g}_0$, this similarity reduction can be expressed as

$$\begin{aligned} &\partial_1(U_2) - \partial_2(U_1) + [U_2, U_1] = 0, \\ &[\Lambda_1, U_2] - [\Lambda_2, U_1] = 0, \\ &t_1\partial_1(U_k) + t_2\partial_2(U_k) + U_k = 0 \quad (k = 1, 2). \end{aligned} \tag{3.6}$$

Note that the operators $\Lambda_k \in \mathfrak{g}_1$ are given by

$$\Lambda_1 = e_1 + 2e_3 + e_5 + e_{21} + e_{60} + e_{23} + e_{43} + e_{63} + e_{234} + e_{236} + e_{436} + 2e_{6234},$$
$$\Lambda_2 = 2e_0 - 2e_3 - 2e_5 - 2e_{21} - 2e_{45} + 2e_{23} + 2e_{43} - 7e_{63} - 4e_{234} + 5e_{236} - 4e_{436} - 2e_{6234}.$$

In the following, we use the notation of a $\mathfrak{g}_{\geqslant 0}$-valued 1-form $\mathcal{B} = B_1\,\mathrm{d}t_1 + B_2\,\mathrm{d}t_2$ with respect to the coordinates $\boldsymbol{t} = (t_1, t_2)$. Then the similarity reduction (3.5) is expressed as

$$d_{\boldsymbol{t}}\mathcal{M} = [\mathcal{B}, \mathcal{M}], \qquad d_{\boldsymbol{t}}\mathcal{B} = \mathcal{B} \wedge \mathcal{B}, \tag{3.7}$$

where $d_{\boldsymbol{t}}$ stands for an exterior differentiation with respect to $\boldsymbol{t}$. Denoting by

$$\mathcal{M}_1 = -t_1\Lambda_1 - t_2\Lambda_2, \qquad \mathcal{B}_1 = \Lambda_1\,\mathrm{d}t_1 + \Lambda_2\,\mathrm{d}t_2,$$

we can express the operators $\mathcal{M}$ and $\mathcal{B}$ in the form

$$\mathcal{M} = \theta + \sum_{i=2,4,6} \xi_i e_i + \sum_{i=2,4,6} \psi_i f_i + \mathcal{M}_1,$$
$$\mathcal{B} = \boldsymbol{u} + \sum_{i=2,4,6} \boldsymbol{x}_i e_i + \sum_{i=2,4,6} \boldsymbol{y}_i f_i + \mathcal{B}_1,$$

where

$$\theta = \vartheta + \sum_{i=0}^{6} \theta_i \alpha_i^\vee, \qquad \boldsymbol{u} = \sum_{i=0}^{6} \boldsymbol{u}_i \alpha_i^\vee.$$

The system (3.7) is expressed in terms of these variables as follows:

$$d_{\boldsymbol{t}}\theta_i = \boldsymbol{x}_i\psi_i - \boldsymbol{y}_i\xi_i, \quad d_{\boldsymbol{t}}\theta_j = 0,$$
$$d_{\boldsymbol{t}}\xi_i = \left(\boldsymbol{u}\middle|\alpha_i^\vee\right)\xi_i - \boldsymbol{x}_i\left(\theta\middle|\alpha_i^\vee\right),$$
$$d_{\boldsymbol{t}}\psi_i = -\left(\boldsymbol{u}\middle|\alpha_i^\vee\right)\psi_i + \boldsymbol{y}_i\left(\theta\middle|\alpha_i^\vee\right)$$

and

$$d_{\boldsymbol{t}}\boldsymbol{u}_i = \boldsymbol{x}_i \wedge \boldsymbol{y}_i + \boldsymbol{y}_i \wedge \boldsymbol{x}_i, \qquad d_{\boldsymbol{t}}\boldsymbol{u}_j = 0,$$
$$d_{\boldsymbol{t}}\boldsymbol{x}_i = \left(\boldsymbol{u}\middle|\alpha_i^\vee\right) \wedge \boldsymbol{x}_i, \qquad d_{\boldsymbol{t}}\boldsymbol{y}_i = -\left(\boldsymbol{u}\middle|\alpha_i^\vee\right) \wedge \boldsymbol{y}_i,$$

for $i = 2, 4, 6$ and $j = 0, 1, 3, 5$.

In this section, we proposed three representations (3.5), (3.6) and (3.7) of the similarity reduction. In the following, we use the system (3.7) in order to derive the system (1.2).

4. Derivation of coupled P_{VI}

In this section, we derive the Hamiltonian system (1.2) from the similarity reduction (3.7). Let $\mathfrak{n}_+$ be the subalgebra of $\mathfrak{g}$ generated by $e_i (i = 0, \ldots, 6)$ and $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ the Borel subalgebra of $\mathfrak{g}$. We introduce below a gauge transformation for the system (3.7)

$$\mathcal{M}^+ = \exp(\mathrm{ad}(\Gamma))\mathcal{M}, \qquad d_{\boldsymbol{t}} - \mathcal{B}^+ = \exp(\mathrm{ad}(\Gamma))(d_{\boldsymbol{t}} - \mathcal{B}),$$

with $\Gamma \in \mathfrak{g}_0$ such that $\mathcal{M}^+$ and $\mathcal{B}^+$ should take values in $\mathfrak{b}_+$.

We first consider a gauge transformation

$$\mathcal{M}^* = \exp(\mathrm{ad}(\Gamma_1))\mathcal{M}, \qquad d_{\boldsymbol{t}} - \mathcal{B}^* = \exp(\mathrm{ad}(\Gamma_1))(d_{\boldsymbol{t}} - \mathcal{B}),$$

with $\Gamma_1 \in \mathfrak{g}_0 \cap \mathfrak{b}_+$ such that

$$\exp(\mathrm{ad}(\Gamma_1))(\mathcal{M}_1) = \sum_{i=0,1,3,5} c_i e_i + e_{21} + e_{45} + e_{60} + e_{23} + e_{43} + c_{63}e_{63} + e_{234}.$$

Note that c_0, c_1, c_3, c_5 and c_{63} are algebraic functions in t_1 and t_2. Then we have

$$d_t\mathcal{M}^* = [\mathcal{B}^*, \mathcal{M}^*], \qquad d_t\mathcal{B}^* = \mathcal{B}^* \wedge \mathcal{B}^*. \tag{4.1}$$

With the notation

$$\mathcal{M}_1^* = \exp(\mathrm{ad}(\Gamma_1))(\mathcal{M}_1), \qquad \mathcal{B}_1^* = \exp(\mathrm{ad}(\Gamma_1))(\mathcal{B}_1),$$

the operators $\mathcal{M}^*$ and $\mathcal{B}^*$ are expressed in the form

$$\mathcal{M}^* = \theta^* + \sum_{i=2,4,6} \xi_i^* e_i + \sum_{i=2,4,6} \psi_i^* f_i + \mathcal{M}_1^*,$$
$$\mathcal{B}^* = \boldsymbol{u}^* + \sum_{i=2,4,6} \boldsymbol{x}_i^* e_i + \sum_{i=2,4,6} \boldsymbol{y}_i^* f_i + \mathcal{B}_1^*,$$

where

$$\theta^* = \vartheta + \sum_{i=0}^{6} \theta_i^* \alpha_i^\vee, \qquad \boldsymbol{u}^* = \sum_{i=0}^{6} \boldsymbol{u}_i^* \alpha_i^\vee.$$

We next consider a gauge transformation

$$\mathcal{M}^+ = \exp(\mathrm{ad}(\Gamma_2))\mathcal{M}^*, \qquad d_t - \mathcal{B}^+ = \exp(\mathrm{ad}(\Gamma_2))(d_t - \mathcal{B}^*),$$

with $\Gamma_2 = \sum_{i=2,4,6} \lambda_i f_i$ such that $\mathcal{M}^+, \mathcal{B}^+ \in \mathfrak{b}_+$, namely

$$\xi_i^*\lambda_i^2 - \left(\theta^* \middle| \alpha_i^\vee\right)\lambda_i - \psi_i^* = 0 \quad (i = 2, 4, 6) \tag{4.2}$$

and

$$d_t\lambda_i = \boldsymbol{x}_i^*\lambda_i^2 - \left(\boldsymbol{u}^* \middle| \alpha_i^\vee\right)\lambda_i - \boldsymbol{y}_i^* \quad (i = 2, 4, 6). \tag{4.3}$$

Here we have

Lemma 4.1. *Under the system (4.1), equation (4.3) follows from equation (4.2).*

Proof. The system (4.1) can be expressed as

$$\begin{aligned} &d_t\theta_i^* = \boldsymbol{x}_i^*\psi_i^* - \boldsymbol{y}_i^*\xi_i^*, \quad d_t\theta_j^* = 0, \\ &d_t\xi_i^* = \left(\boldsymbol{u}^* \middle| \alpha_i^\vee\right)\xi_i^* - \boldsymbol{x}_i^*\left(\theta^* \middle| \alpha_i^\vee\right), \\ &d_t\psi_i^* = -\left(\boldsymbol{u}^* \middle| \alpha_i^\vee\right)\psi_i^* + \boldsymbol{y}_i^*\left(\theta^* \middle| \alpha_i^\vee\right), \end{aligned} \tag{4.4}$$

for $i = 2, 4, 6$ and $j = 0, 1, 3, 5$. By using (4.4) and $(d_t\theta^*|\alpha_i^\vee) = 2d_t\theta_i^*$, we obtain

$$d_t\left(\xi_i^*\lambda_i^2 - \left(\theta^* \middle| \alpha_i^\vee\right)\lambda_i - \psi_i^*\right) = \left\{2\xi_i^*\lambda_i - (\theta^*|\alpha_i^\vee)\right\}\left\{d_t\lambda_i - \boldsymbol{x}_i^*\lambda_i^2 + \left(\boldsymbol{u}^* \middle| \alpha_i^\vee\right)\lambda_i + \boldsymbol{y}_i^*\right\}$$
$$(i = 2, 4, 6).$$

It follows that equation (4.2) implies (4.3) or

$$\lambda_i = \frac{\left(\theta^* \middle| \alpha_i^\vee\right)}{2\xi_i^*} \quad (i = 2, 4, 6). \tag{4.5}$$

Hence, it is enough to verify that equation (4.3) follows from (4.5). Together with (4.4), equation (4.5) implies

$$\begin{aligned} d_t\lambda_i &= \frac{\left(d_t\theta^* \middle| \alpha_i^\vee\right)\xi_i^* - \left(\theta^* \middle| \alpha_i^\vee\right)d_t\xi_i^*}{2(\xi_i^*)^2} \\ &= \boldsymbol{x}_i^*\lambda_i^2 - \left(\boldsymbol{u}^* \middle| \alpha_i^\vee\right)\lambda_i - \boldsymbol{y}_i^* + \frac{\boldsymbol{x}_i^*\left\{4\xi_i^*\psi_i^* + \left(\theta^* \middle| \alpha_i^\vee\right)^2\right\}}{4(\xi_i^*)^2}. \end{aligned} \tag{4.6}$$

On the other hand, we obtain

$$4\xi_i^*\psi_i^* + \left(\theta^*\middle|\alpha_i^\vee\right)^2 = 0 \tag{4.7}$$

by substituting (4.5) into (4.2). Combining (4.6) and (4.7), we obtain equation (4.3). □

Thanks to lemma 4.1, the gauge parameters $\lambda_i (i = 2, 4, 6)$ are determined by equation (4.2). Hence we obtain the system on $\mathfrak{b}_+$

$$d_t\mathcal{M}^+ = [\mathcal{B}^+, \mathcal{M}^+], \qquad d_t\mathcal{B}^+ = \mathcal{B}^+ \wedge \mathcal{B}^+, \tag{4.8}$$

with dependent variables λ_i and $\mu_i = \xi_i^* (i = 2, 4, 6)$. The operator $\mathcal{M}^+$ is described as

$$\mathcal{M}^+ = \kappa + \sum_{i=2,4,6} \mu_i e_i + (c_0 + \lambda_6)e_0 + (c_1 + \lambda_2)e_1 + (c_3 + \lambda_2 + \lambda_4 + c_{63}\lambda_6 - \lambda_2\lambda_4)e_3$$
$$+ (c_5 + \lambda_4)e_5 + e_{21} + e_{45} + e_{60} + (1 - \lambda_4)e_{23} + (1 - \lambda_2)e_{43} + c_{63}e_{63} + e_{234},$$

where $\kappa \in \mathfrak{h}$. Note that $d_t\kappa = 0$.

Let s_1 and s_2 be independent variables defined by

$$s_1 = \frac{c_{63}(1 + c_3 - c_0c_{63})}{6}, \qquad s_2 = \frac{c_{63}(1 + c_1)(1 + c_5)}{6}.$$

We now regard the system (4.8) as a system of ordinary differential equations

$$\left[s(s-1)\frac{\mathrm{d}}{\mathrm{d}s} - B, \mathcal{M}^+\right] = 0, \tag{4.9}$$

with respect to the independent variable $s = s_1$ by setting $s_2 = 1$. The operator B is expressed in the form

$$B = \sum_{i=0}^{6} u_i\alpha_i^\vee + \sum_{i=0}^{6} x_ie_i + x_{21}e_{21} + x_{45}e_{45} + x_{23}e_{23} + x_{43}e_{43}$$
$$+ x_{63}e_{63} + x_{234}e_{234} + x_{236}e_{236} + x_{436}e_{436} + x_{6234}e_{6234}.$$

Each coefficient of B is a polynomial in λ_i and μ_i; we do not give the explicit formula.

Let $q_i, p_i (i = 1, 2, 3)$ be dependent variables defined by

$$q_1 = \frac{1 - \lambda_2}{1 + c_1}, \qquad q_2 = \frac{1 - \lambda_4}{1 + c_5}, \qquad q_3 = \frac{1 + c_3 - c_0c_{63}}{1 + c_3 + c_{63}\lambda_6},$$
$$p_1 = -\frac{(1 + c_1)\mu_2}{6}, \qquad p_2 = -\frac{(1 + c_5)\mu_4}{6}, \tag{4.10}$$
$$p_3 = -\frac{(1 + c_3 + c_{63}\lambda_6)\left\{(1 + c_3 + c_{63}\lambda_6)\mu_6 + c_{63}\left(\kappa\middle|\alpha_6^\vee\right)\right\}}{6c_{63}(1 + c_3 - c_0c_{63})}.$$

We also set

$$\alpha_i = \frac{\left(\kappa\middle|\alpha_i^\vee\right)}{6} \quad (i = 0, \ldots, 6).$$

Then we obtain

Theorem 4.2. *The system (4.9) is equivalent to the system (1.2) with (1.1).*

Remark 4.3. The system (1.2) with (1.1) can be regarded as the compatibility condition of a Lax pair

$$\mathcal{M}^+\boldsymbol{w} = 0, \qquad s(s-1)\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}s} = B\boldsymbol{w}, \tag{4.11}$$

where $\boldsymbol{w} = \exp(\Gamma)W \exp\left(\sum_{k=1}^{\infty} t_k \Lambda_k\right)$. On the other hand, the affine Lie algebra $\mathfrak{g}\left(E_6^{(1)}\right)$ is realized as a central extension of the loop algebra $\mathfrak{g}(E_6)[z, z^{-1}]$ with a derivation $z\mathrm{d}/\mathrm{d}z$. In this framework, the system (4.11) can be identified with a Lax pair

$$z\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}z} = M\boldsymbol{w}, \qquad s(s-1)\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}s} = B\boldsymbol{w},$$

where $M = (6d - \mathcal{M}^{+})/6$.

Lastly, we note a derivation of the affine Weyl group symmetry for the system (1.2). We define a Poisson structure for the $\mathfrak{b}_{+}$-valued operator $\mathcal{M}^{+}$ by

$$\{\mu_i, \lambda_j\} = 6\delta_{i,j}, \quad \{\mu_i, \mu_j\} = \{\lambda_i, \lambda_j\} = 0 \quad (i, j = 2, 4, 6).$$

It is equivalent to

$$\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \quad (i, j = 1, 2, 3),$$

via the transformation (4.10). Hence $p_i, q_i (i = 1, 2, 3)$ give a canonical coordinate system associated with the Poisson structure for $\mathcal{M}^{+}$.

Thanks to [NY2], we then obtain birational canonical transformations $r_i (i = 0, \ldots, 6)$ given in theorem 1.1. They are derived from the transformations

$$r_i(X) = X\exp(-e_i)\exp(f_i)\exp(-e_i) \quad (i = 0, \ldots, 6),$$

where $X = \exp\left(\sum_{k=1}^{\infty} t_k \Lambda_k\right) X(0)$.

Acknowledgments

The authors are grateful to Professors Masatoshi Noumi, Yasuhiko Yamada and Teruhisa Tsuda for valuable discussions and advice. This work is partially supported by a fellowship of the Japan Society for the Promotion of Science (JSPS).

References

[AS] Ablowitz M J and Segur H 1977 Exact linearization of a Painlevé transcendent *Phys. Rev. Lett.* **38** 1103–6
[C] Carter R 1972 Conjugacy classes in the Weyl group *Compos. Math.* **25** 1–59
[DF] Delduc F and Fehér L 1995 Regular conjugacy classes in the Weyl group and integral hierarchies *J. Phys. A: Math. Gen.* **28** 5843–82
[DS] Drinfel'd V G and Sokolov V V 1985 Lie algebras and equations of Korteweg–de Vries type *J. Sov. Math.* **30** 1975–2036
[FS1] Fuji K and Suzuki T 2006 The sixth Painlevé equation arising from $D_4^{(1)}$ hierarchy *J. Phys. A: Math. Gen.* **39** 12073–82
[FS2] Fuji K and Suzuki T 2008 Higher order Painlevé system of type $D_{2n+2}^{(1)}$ arising from integrable hierarchy *Int. Math. Res. Not.* **1** 1–21
[IKSY] Iwasaki K, Kimura H, Shimomura S and Yoshida M 1991 From Gauss to Painlevé—a modern theory of special functions *Aspects of Mathematics* vol E16 (Braunschweig: Vieweg)
[Kac] Kac V G 1990 *Infinite Dimensional Lie Algebras* (Cambridge: Cambridge University Press)
[KIK] Kikuchi T, Ikeda T and Kakei S 2003 Similarity reduction of the modified Yajima–Oikawa equation *J. Phys. A: Math. Gen.* **36** 11465–80
[KK1] Kakei S and Kikuchi T 2004 Affine Lie group approach to a derivative nonlinear Schrödinger equation and its similarity reduction *Int. Math. Res. Not.* **78** 4181–209
[KK2] Kakei S and Kikuchi T 2007 The sixth Painlevé equation as similarity reduction of $\widehat{\mathfrak{gl}}_3$ hierarchy *Lett. Math. Phys.* **79** 221–34
[KP] Kac V G and Peterson D 1985 112 Constructions of the basic representation of the Roop Group of E_8 *Symp. on Anomalies, Geometry and Topology* ed W A Baedeen and A R White (Singapore: World Scientific) pp 276–98
[NY1] Noumi M and Yamada Y 1998 Higher order Painlevé equations of type $A_l^{(1)}$ *Funkcial. Ekvac.* **41** 483–503

[NY2] Noumi M and Yamada Y 2001 Birational Weyl group action arising from a nilpotent Poisson algebra *Physics and Combinatorics 1999 Proc. Nagoya 1999 Int. Workshop* ed A N Kirillov, A Tsuchiya and H Umemura (Singapore: World Scientific) pp 287–319

[O1] Okamoto K 1987 Studies on the Painlevé equations: I *Ann. Math. Pura Appl.* **146** 337–81
Okamoto K 1987 Studies on the Painlevé equations: II *Japan. J. Math.* **13** 47–76
Okamoto K 1986 Studies on the Painlevé equations: III *Math. Ann.* **275** 221–56
Okamoto K 1987 Studies on the Painlevé equations: IV *Funkcial. Ekvac.* **30** 305–32

[O2] Okamoto K 1999 The Hamiltonians associated with the Painlevé equations *The Painlevé Property: One Century Later (CRM Series in Mathematical Physics)* ed R Conte (Berlin: Springer)

[S] Sasano Y 2006 Higher-order Painlevé equations of type $D_l^{(1)}$ *RIMS Koukyuroku* **1473** 143–63